V. *Corrections and Additions to the Memoir on the Theory of Reciprocal Surfaces* (Philosophical Transactions, vol. clix. 1869). *By Professor* CAYLEY, *F.R.S.*

Received July 22,—Read November 16, 1871.

1. I AM indebted to Dr. ZEUTHEN for the remark that although the "off-points" and "off-planes," as explained in the memoir, are real singularities, they are not the singularities to which the θ, θ' of the formulæ refer. The most convenient way of correcting this is to retain all the formulæ with θ, θ' as they stand, but to write ω, ω' for the number of "off-points" and "off-planes" respectively; viz. we thus have

ω, off-points,
θ, unexplained singular points,

and

ω', off-planes,
θ', unexplained singular planes,

the formulæ as they stand, taking account of the unexplained singularities θ and θ', but not taking any account at all of the off-points and off-planes ω, ω'. The extended formulæ in which these are taken into account are:—

$$\begin{aligned}
a(n-2)&=\varkappa-\mathrm{B}+\varrho+2\sigma+3\omega,\\
b(n-2)&=\varrho+2\beta+3\gamma+3t,\\
c(n-2)&=2\sigma+4\beta+\gamma+\theta+\omega,\\
a(n-2)(n-3)&=2(\delta-\mathrm{C}-3\omega)+3(ac-3\sigma-\chi-3\omega)+2(ab-2\varrho-j),\\
b(n-2)(n-3)&=4k+(ab-2\varrho-j)+3(bc-3\beta-2\gamma-i),\\
c(n-2)(n-3)&=6h+(ac-3\sigma-\chi-3\omega)+2(bc-3\beta-2\gamma-i),
\end{aligned}$$

which replace SALMON's original formulæ (A) and (B).

2. In the formulæ

$$\begin{aligned}
q&=b^2-b-2k-3\gamma-6t,\\
r&=c^2-c-2h-3\beta,
\end{aligned}$$

it is assumed that the nodal curve has no actual multiple points other than the t triple points, and no stationary points other than the γ points which lie on the cuspidal curve; and similarly that the cuspidal curve has no actual multiple points, and no stationary points other than the β points which lie on the nodal curve; and this being so, q is the class of the nodal curve and r that of the cuspidal curve. But we may take the formulæ as *universally* true; viz. q may be considered as standing for $b^2-b-2k-3\gamma-6t$, and r

as standing for $c^2-c-2h-3\beta$; only then q and r are not in all cases the classes of the two curves respectively.

3. In the formulæ No. 6 *et seq.*, introducing the new singularity ω, we have as follows:—

$$(a-b-c)(n-2) \qquad =(\varkappa-\mathrm{B}-\theta+2\omega)-6\beta-4\gamma-3t,$$
$$(a-2b-3c)(n-2)(n-3)=2(\delta-\mathrm{C}-3\omega)-8k-18h-12(bc-3\beta-2\gamma-i);$$

and substituting these in $n'=a(a-1)-2b-3c$, and writing for n' its value $=a(a-1)-2\delta-3\varkappa$, we have, as in the memoir,

$$\begin{aligned} n'=n(n-1)^2-n(7b+12c)+4b^2+8b+9c^2+15c \\ -8k-8h+18\beta+12\gamma+12i-9t \\ -2\mathrm{C}-3\mathrm{B}-3\theta; \end{aligned}$$

viz. there is no term in ω.

Writing $(n-2)(n-3)=a+2b+3c+(-4n+6)$ in the equations which contain $(n-2)(n-3)$, these become

$$\begin{aligned} a(-4n+6)&=2(\delta-\mathrm{C})-\ a^2-4\varrho-9\sigma-2j-3\chi-15\omega, \\ b(-4n+6)&=\ 4k\ \ -2b^2-9\beta-6\gamma-3i-2\varrho-j, \\ c(-4n+6)&=\ 6h\ \ -3c^2-6\beta-4\gamma-2i-3\sigma-\chi-3\omega, \end{aligned}$$

(SALMON's equations (C)); and adding to each equation four times the corresponding equation with the factor $(n-2)$, these become

$$\begin{aligned} a^2-2a&=2(\delta-\mathrm{C})+4(\varkappa-\mathrm{B})-\sigma-2j-3\chi-3\omega, \\ 2b^2-2b&=4k-\beta+6\gamma+12t-3i+2\varrho-j, \\ 3c^2-2c&=6h+10\beta+4\theta-2i+5\sigma-\chi+\omega. \end{aligned}$$

Writing in the first of these $a^2-2a=n'+2\delta+3\varkappa-a$, and reducing the other two by means of the values of q, r, the equations become

$$\begin{aligned} n'-a&=-2\mathrm{C}-4\mathrm{B}+\varkappa-\sigma-2j-3\chi-3\omega, \\ 2q+\beta+3i+j&=2\varrho, \\ 3r+c+2i+\chi&=5\sigma+\beta+4\theta+\omega. \end{aligned}$$

The reciprocal of the first of these is

$$\sigma'=a-n+\varkappa'-2j'-3\chi'-2\mathrm{C}'-4\mathrm{B}'-3\omega';$$

viz. writing $a=n(n-1)-2b-3c$, and $\varkappa=3n(n-2)-6b-8c$, this is

$$\sigma'=4n(n-2)-8b-11c-2j'-3\chi'-2\mathrm{C}'-4\mathrm{B}'-3\omega';$$

and it thus appears that the order σ' of the spinode curve is reduced by 3 for each off-plane ω'.

4. As to the other two equations, writing for ϱ, σ their values, these become

$$j+6t+3i+5\beta+6\gamma=b(2n-4)-2q,$$
$$2\chi+3\omega+4i+18\beta+5\gamma=c(5n-12)-6r+3\theta,$$

equations which admit of a geometrical interpretation. In fact, when there is only a nodal curve, the first equation is

$$j+6t=b(2n-4)-2q,$$

which we may verify when the nodal curve is a complete intersection, $P=0$, $Q=0$; for if the equation of the surface is $(A, B, C \between P, Q)^2=0$, where the degrees of A, B, C, P, Q are $n-2f$, $n-f-g$, $n-2g$, f, g respectively, then the pinch-points are given by the equations $P=0$, $Q=0$, $AC-B^2=0$, and the number j of pinch-points is thus

$$=fg(2n-2f-2g)=(2n-4)fg-2fg(f+g-2);$$

but for the curve $P=0$, $Q=0$ we have $t=0$, and its order and class are $b=fg$, $q=fg(f+g-2)$, or the formula is thus verified.

Similarly, when there is only a cuspidal curve, the second equation is

$$2\chi+3\omega=c(5n-12)-6r+3\theta,$$

which may be verified when the cuspidal curve is a complete intersection, $P=0$, $Q=0$; the equation of the surface is here $(A, B, C \between P, Q)^2=0$, where $AC-B^2=MP+NQ$, and the points χ, ω are given as the intersections of the curve with the surface $(A, B, C \between N, -M)^2=0$.

Now $AC-B^2$ vanishing for $P=0$, $Q=0$ we must have $A=\Lambda\alpha^2+A'$, $B=\Lambda\alpha\beta+B'$, $C=\Lambda\beta^2+C'$, where A', B', C' vanish for $P=0$, $Q=0$; and thence $M=\Lambda M'+M''$, $N=\Lambda N'+N''$, where M'', N'' vanish for $P=0$, $Q=0$. The equation

$$(A, B, C \between N, -M)^2=0,$$

writing therein $P=0, Q=0$, thus becomes $\Lambda^3(N'\alpha-M'\beta)^2=0$; and its intersections with the curve $P=0, Q=0$ are the points $P=0$, $Q=0$, $\Lambda=0$ each three times, and the points $P=0$, $Q=0$, $N'\alpha-M'\alpha=0$ each twice; viz. they are the points $2\chi+3\omega$.

But if the degree of Λ is $=\lambda$, then the degrees of N', M', α^2, $\alpha\beta$, β^2 are $2n-3f-2g-\lambda$, $2n-2f-3g-\lambda$, $n-2f-\lambda$, $n-f-g-\lambda$, $n-2g-\lambda$, whence the degree of $\Lambda^3(N'\alpha-M'\beta)$ is $=5n-6f-6g$, and the number of points is $=fg(5n-6f-6g)$, viz. this is

$$=fg(5n-12)-6fg(f+g-2),$$

or it is $=c(5n-12)-6r$; so that θ being $=0$, the equation is verified.

5. It was also pointed out to me by Dr. ZEUTHEN that in the value of $24t$ given in No. 10 the term involving χ should be -6χ instead of $+6\chi$, and that in consequence the coefficients of χ are erroneous in several others of the formulæ. Correcting these,

and at the same time introducing the terms in ω, and writing down also the terms in θ as they stand, we have

$$\begin{aligned}
4i &= \ldots - 2\chi + 3\theta - 3\omega,\\
24t &= \ldots - 6\chi + 9\theta - 9\omega,\\
2\sigma &= \ldots \qquad - \theta - \omega,\\
8\varrho &= \ldots + 6\chi - 9\theta + 9\omega,\\
8\varkappa &= \ldots - 6\chi + 17\theta - 25\omega,\\
2\delta &= \ldots + 6\chi - 9\theta + 15\omega,\\
8n' &= \ldots - 30\chi + 21\theta - 45\omega,\\
c' &= \ldots - 12\chi + 10\theta - 20\omega.
\end{aligned}$$

The equations of No. 11, used afterwards, No. 53, should thus be

$$\begin{aligned}
4i + 6r &= (5n-12)c - 18\beta - 5\gamma - 2\chi + 3\theta - 3\omega,\\
-24t - 8q + 18r &= (-8n+16)b + (15n-36)c - 34\beta + 9\gamma + 4j - 6\chi + 9\theta - 9\omega;
\end{aligned}$$

and from these I deduce

$$44q + \tfrac{63}{2}r = (44n-88)b + (\tfrac{105}{4}n - 63)c - \tfrac{409}{2}\beta - \tfrac{633}{4}\gamma - 132t - 87i - 22j - \tfrac{21}{2}\chi + \tfrac{63}{4}\theta.$$

6. In No. 32 we have (without alteration) $\theta=16$; but in the application (Nos. 40 and 41) to the surface $FP^2+GR^2Q^3=0$ we have $\theta=0$, and there are $\omega=fpq$ off-points, $F=0$, $P=0$, $Q=0$, and $\chi=gpq$ close-points, $G=0$, $P=0$, $Q=0$. The new equations involving ω are thus satisfied.

7. I have ascertained that the value of β' obtained, Nos. 51 to 64 of the memoir, is inconsistent with that obtained in the "Addition" by consideration of the deficiency, and that it is in fact incorrect. The reason is that, although, as stated No. 53, the values of two of the coefficients D, E may be assumed at pleasure, they cannot, in conjunction with a given system of values of A, B, C, be thus assumed at pleasure; viz. A, B, C being $=110$, 272, 44 respectively, the values of D, E are really determinate. I have no direct investigation, but by working back from the formula in the Addition I find that we must have $D=\frac{477}{4}$, $E=315$; the values of the remaining coefficients then are

$$F=\tfrac{63}{2},\ G=-\tfrac{715}{2},\ H=-\tfrac{1005}{4},\ I=-198;$$

or the formula is

$$\begin{aligned}
\beta' = {} & 2n(n-2)(11n-24)\\
& -(110n-272)b + 44q\\
& -(\tfrac{477}{4}n - 315)c + \tfrac{63}{2}r\\
& +\tfrac{715}{2}\beta + \tfrac{1005}{4}\gamma + 198t\\
& -hC - gB - xi - \lambda j - \mu\chi - \nu\theta - f\omega\\
& -h'C' - g'B' - x'i' - \lambda'j' - \mu'\chi' - \nu'\theta' - f'\omega';
\end{aligned}$$

but I have not as yet any means of determining the coefficients f, f' of the terms in ω, ω'.

From the several cases of a cubic surface we obtain as in the memoir; but applying to the same surfaces the reciprocal equation for β, instead of the results of the memoir, we find

$$\begin{aligned} h' &= -\ \ 4, \\ g'+16\nu &= -198, \\ g'+\ 2\mu &= \ \ 45, \\ g+g' &= \ \ 18, \\ \lambda &= \ \ 5 \end{aligned}$$

(so that now $\lambda+\lambda'=-2$, as is also given by the cubic scroll). And combining the two sets of results, we have

$$\begin{aligned} h &= 24, \\ \lambda &= 5, \\ \mu &= \tfrac{27}{2}+\tfrac{1}{2}g, \\ \nu &= -\tfrac{27}{2}+\tfrac{1}{16}g, \\ h' &= -4, \\ g' &= 18-g, \\ \lambda' &= -7, \\ \mu' &= 6-\tfrac{1}{2}g, \\ \nu' &= \tfrac{9}{4}-\tfrac{1}{16}g; \end{aligned}$$

but the coefficients g, x, x', f, f' are still undetermined. To make the result agree with that of the Addition, I assume $x=-86$, $x'=-1$, $g=+28$; whence we have

$$\begin{aligned} \beta' = {} & 2n(n-2)(11n-24) \\ & -(110n-272)b+44q \\ & -(\tfrac{477}{4}n-315)c+\tfrac{63}{2}r \\ & +\tfrac{715}{2}\beta+\tfrac{1005}{4}\gamma+198t \\ & -24\mathrm{C}-28\mathrm{B}+86i-5j-\tfrac{55}{2}\chi+\tfrac{47}{4}\theta-f\omega \\ & +\ 4\mathrm{C}'+10\mathrm{B}'+i'+7j'+8\chi'-\tfrac{1}{2}\theta'-f'\omega'; \end{aligned}$$

and if we substitute herein the foregoing value of $44q+\frac{63}{2}r$, we obtain

$$\begin{aligned} \beta' = {} & 2n(n-2)(11n-24) \\ & +(-66n+184)b \\ & +(-93n+252)c \\ & +153\beta+93\gamma+66t \\ & -24\mathrm{C}-28\mathrm{B}-i-27j-38\chi+\tfrac{55}{2}\theta-f\omega \\ & +\ 4\mathrm{C}'+10\mathrm{B}'+i'+\ 7j'+\ 8\chi'-\tfrac{1}{2}\theta'-f'\omega', \end{aligned}$$

which, except as to the terms in ω, ω', the coefficients of which are not determined, agrees with the value given in the Addition.

Dr. ZEUTHEN considers that in general $i'=i$; I presume this is so, but have not verified it.